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Rationalized evaluation subgroups of a map I: Sullivan models, derivations and G-sequences

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Abstract

We identify the homomorphism induced on rational homotopy groups by the evaluation map $\omega\colon \mathrm{map}(X, Y; f) \to Y$ in terms of a map of complexes of derivations constructed directly from the Sullivan minimal model of f. This allows us to characterize the rationalized nth evaluation subgroup of f and, more generally, the rationalization of the so-called G-sequence of the map f. We use these results to study the G-sequence in the context of rational homotopy theory.

MSC: 55P62; 55Q52

1. Introduction

Let $f\colon X \to Y$ be a based map of simply connected CW complexes. Denote by $\mathrm{map}(X, Y; f)$ the path component of the space of (unbased) maps $X \to Y$ that consists of maps (freely) homotopic to f. Evaluation at the basepoint of X gives the *evaluation map* $\omega\colon \mathrm{map}(X, Y; f) \to Y$. The *nth evaluation subgroup of* f, denoted $G_n(Y, X; f)$, is the image of $\omega_\#$ in $\pi_n(Y)$. The Gottlieb group $G_n(X)$ occurs as the special case in which $X = Y$ and $f = 1_X$ [7].

Because $\omega\colon \mathrm{map}(X, X; 1) \to X$ may be identified with the connecting map of the universal fibration for fibrations with fibre X, the Gottlieb groups are important universal objects for fibrations with fibre X that feature in a broad area of research in homotopy theory. Some general results about $G_*(X)$ are known but explicit computation is limited to sporadic examples. A particular difficulty is the fact that a map $f\colon X \to Y$ does not induce a corresponding homomorphism of Gottlieb groups, since in general $f_\#(G_n(X)) \not\subseteq G_n(Y)$. On the other hand, $f\colon X \to Y$ always induces a map $f_\#\colon G_n(X) \to G_n(Y, X; f)$. In [24], Lee and Woo observed that this induced homomorphism fits into a larger framework called the *G-sequence of* f, which is the image of the long exact homotopy sequence of $f_*\colon \mathrm{map}(X, X; 1) \to \mathrm{map}(X, Y; f)$ in that of f, under the homomorphisms induced by the appropriate evaluation

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maps. It is pictured as the middle row in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(f_*) \arrow[r] \arrow[d, "\omega_\#"] & \pi_n(\mathrm{map}(X,X;1)) \arrow[r, "(f_*)_\#"] \arrow[d, "\omega_\#"] & \pi_n(\mathrm{map}(X,Y;f)) \arrow[r] \arrow[d, "\omega_\#"] & \cdots \\
\cdots \arrow[r] & G^{\mathrm{rel}}_{n+1}(Y,X;f) \arrow[r] \arrow[d, hook] & G_n(X) \arrow[r, "f_\#"] \arrow[d, hook] & G_n(Y,X;f) \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & \pi_{n+1}(f) \arrow[r] & \pi_n(X) \arrow[r, "f_\#"] & \pi_n(Y) \arrow[r] & \cdots
\end{tikzcd}$$

The G-sequence forms a chain complex, that is, consecutive compositions are trivial, since its homomorphisms are restrictions of those in the long exact homotopy sequence of the map f. Some general conditions are known under which the G-sequence is exact (e.g. [12,19]), but in general it is not exact (e.g. [12]).

Here we bring the techniques of rational homotopy theory to bear on these topics. Theorem 2.1 identifies the map induced on rational homotopy groups by ω with the map induced on homology by a certain map of complexes of derivations of the Sullivan minimal models of X and Y. The basic idea for this identification may be traced back to the paper of Thom [23]. Thom's work, in turn, served as the starting point for the Sullivan–Haefliger model for $\mathrm{map}(X, Y; f)$ sketched by Sullivan in [22] and completed by Haefliger in [9]. The basic idea may be indicated as follows: a map $f: S^n \times X \to Y$ yields a certain morphism of degree $(-n)$ from the Sullivan minimal model of Y to that of X, and this morphism turns out to be a derivation in a generalized sense. The space of these derivations can be endowed with a differential, and the homology of this chain complex yields the rational homotopy groups of the space $\mathrm{map}(X, Y; f)$. Corollary 2.2 describes the rationalized evaluation subgroups of a map in similar terms. In Sections 3 and 4 we develop Theorem 2.1 to study the rationalized G-sequence. Finally, in a technical appendix we present a careful proof of a result from DG algebra homotopy theory that is necessary for our proof of Theorem 2.1.

This paper and its sequel [14] are greatly condensed versions of their originals. We are obliged to the editor and the referees who prodded us to keep condensing. The original referee, in particular, did an extremely thorough job that we found most helpful. During the considerable time that these articles have been circulating, new results have appeared: [3] describes the rational homotopy groups of a function space in an essentially similar way to that included in our Theorem 2.1 (although not using *minimal* models); [15] obtains a formula for the rank of the fundamental group of the space $\mathrm{map}(X, Y; f)$ as an extension of Theorem 2.1; [4] gives formulas for Whitehead products in the rational homotopy groups of function spaces and [11] and [6] give new calculations of Gottlieb groups of homogeneous spaces and spheres.

We end this introduction by fixing notation. By *vector space* we mean a non-negatively graded rational vector space. By *DG (differential graded) algebra*, we mean a commutative, differential, non-negatively graded algebra over $\mathbb{Q}$ that is connected ($H^0 = \mathbb{Q}$), simply connected ($H^1 = 0$), and has cohomology of finite type. For a vector space V, we denote by ΛV the free commutative graded algebra generated by V. If f is a map, then f^*, respectively f_*, denotes pre-, respectively post-composition by f. We denote by $H(f)$, respectively $f_\#$, the map induced by f on homology (or cohomology), respectively homotopy.

We assume that the reader is familiar with the basics of rational homotopy theory. Our general reference for this material is [5]. We recall here that a space X has a *minimal model* $(\mathcal{M}_X, \mathrm{d}_X)$, with $\mathcal{M}_X$ a free algebra ΛV and d_X a decomposable differential, that is, $\mathrm{d}_X(V) \subseteq \Lambda^{\geq 2} V$. Furthermore, a map of spaces $f: X \to Y$ induces a map of minimal models $\mathcal{M}_f: \mathcal{M}_Y \to \mathcal{M}_X$. We refer to this induced map as the *Sullivan minimal model* of the map f.

We say two maps of vector spaces $f: U \to V$ and $g: U' \to V'$ are *equivalent* if there exist isomorphisms $\alpha: U \to U'$ and $\beta: V \to V'$ such that $\beta \circ f = g \circ \alpha$. This notion extends in the obvious way to sequences, commutative squares and any other diagram of vector space maps.

If $A^0 \cong \mathbb{Q}$, then the map $\varepsilon: A \to \mathbb{Q}$ that sends all elements of positive degree to zero and is the identity in degree zero, is the unique augmentation of A. We regard $\mathbb{Q}$ as the trivial DG algebra concentrated in degree zero and with trivial differential; thus ε is a DG algebra map. Given DG algebras (A, d_A) and (B, d_B) and a (fixed) DG algebra map $\phi: A \to B$, define a *ϕ-derivation* of degree n to be a linear map $\theta: A \to B$ that *reduces degree by n* and satisfies the derivation law $\theta(xy) = \theta(x)\phi(y) + (-1)^{n|x|}\phi(x)\theta(y)$. We will only consider derivations of positive degree, that is, those that reduce degree by some positive integer. Let $\mathrm{Der}_n(A, B; \phi)$ denote the vector space of ϕ-derivations of degree n, for $n > 0$. Finally, define a linear map $D\colon \mathrm{Der}_n(A, B; \phi) \to \mathrm{Der}_{n-1}(A, B; \phi)$ by $D(\theta) = \mathrm{d}_B \circ \theta - (-1)^{|\theta|}\theta \circ \mathrm{d}_A$. A standard check now shows that $D^2 = 0$ and thus $(\mathrm{Der}_*(A, B; \phi), D)$ is a chain complex. In case $A = B$ and $\phi = 1_B$, the chain complex of derivations $\mathrm{Der}_*(B, B; 1)$ is just the usual complex of derivations on the DG algebra B. In order to cut down on cumbersome notation, we will usually suppress the differential from our notation, and write $H_n(\mathrm{Der}(A, B; \phi))$ for the homology in degree n of the chain complex $(\mathrm{Der}_*(A, B; \phi), D)$. Pre-composition with ϕ, respectively post-composition by the augmentation $\varepsilon\colon B \to \mathbb{Q}$, gives a map of chain complexes $\phi^*\colon \mathrm{Der}_*(B, B; 1) \to \mathrm{Der}_*(A, B; \phi)$, respectively $\varepsilon_*\colon \mathrm{Der}_*(A, B; \phi) \to \mathrm{Der}_*(A, \mathbb{Q}; \varepsilon)$.

2. Rational homotopy of function spaces

Suppose $\mathcal{M}_f\colon \mathcal{M}_Y \to \mathcal{M}_X$ is the Sullivan minimal model of the map f. Then we have the following commutative square of chain complexes:

$$\begin{tikzcd}
\mathrm{Der}_*(\mathcal{M}_X, \mathcal{M}_X; 1) \arrow[r, "(\mathcal{M}_f)^*"] \arrow[d, "\varepsilon_*"'] & \mathrm{Der}_*(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f) \arrow[d, "\varepsilon_*"] \\
\mathrm{Der}_*(\mathcal{M}_X, \mathbb{Q}; \varepsilon) \arrow[r, "\widehat{(\mathcal{M}_f)^*}"] & \mathrm{Der}_*(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)
\end{tikzcd} \tag{1}$$

Both horizontal maps are obtained by pre-composing with the same map $\mathcal{M}_f$, but in different contexts. Since we will need to distinguish between these two maps notationally in the sequel, we have used an extra decoration on the bottom one.

Theorem 2.1. *Let X and Y be simply connected CW complexes of finite type, with X finite. For $n \geq 2$, the commutative square*

$$\begin{tikzcd}
\pi_n(\mathrm{map}(X, X; 1)) \otimes \mathbb{Q} \arrow[r, "(f_*)_\# \otimes 1"] \arrow[d, "\omega_\# \otimes 1"'] & \pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \arrow[d, "\omega_\# \otimes 1"] \\
\pi_n(X) \otimes \mathbb{Q} \arrow[r, "f_\# \otimes 1"'] & \pi_n(Y) \otimes \mathbb{Q}
\end{tikzcd}$$

is equivalent to that obtained from (1) by passing to homology.

Proof. We will define vector space isomorphisms Φ, Φ_f, Ψ_X, and Ψ_Y to give the following equivalence of commutative squares:

$$\begin{tikzcd}
 & H_n(\mathrm{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) \arrow[rr, "H((\mathcal{M}_f)^*)"] \arrow[dd, "H(\varepsilon_*)"] & & H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \arrow[dd, "H(\varepsilon_*)"] \\
\pi_n(\mathrm{map}(X, X; 1)) \otimes \mathbb{Q} \arrow[ur, "\Phi", "\cong"'] \arrow[rr, "(f_*)_\# \otimes 1"'] \arrow[dd, "\omega_\#"'] & & \pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \arrow[ur, "\Phi_f", "\cong"'] \arrow[dd, "\omega_\#"] & \\
 & H_n(\mathrm{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon)) \arrow[rr, "H(\widehat{(\mathcal{M}_f)^*})"'] & & H_n(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)) \\
\pi_n(X) \otimes \mathbb{Q} \arrow[ur, "\Psi_X", "\cong"'] \arrow[rr, "f_\# \otimes 1"'] & & \pi_n(Y) \otimes \mathbb{Q} \arrow[ur, "\Psi_Y", "\cong"'] &
\end{tikzcd}$$

The map Ψ_X corresponds to the standard identification $\pi_*(X) \otimes \mathbb{Q} \cong \mathrm{Hom}(Q(\mathcal{M}_X), \mathbb{Q}) \cong H_*(\mathrm{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon))$ and similarly for Ψ_Y. We obtain Φ_f as the rationalization of a natural homomorphism

$$\Phi'_f\colon\ \pi_n(\mathrm{map}(X, Y; f)) \to H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)),$$

which we now define. The adjoint of a representative of a homotopy class $\alpha \in \pi_n(\mathrm{map}(X, Y; f))$ is a map $F\colon S^n \times X \to Y$ that satisfies $F \circ i_2 = f$, where $i_2\colon X \to S^n \times X$ is the inclusion. We say that F is *a map* *under* f — see the appendix for this and other terminology we use in this proof. Passing to minimal models, we obtain a map $\mathcal{M}_F\colon \mathcal{M}_Y \to \mathcal{M}_{S^n} \otimes \mathcal{M}_X$. Let $p_2\colon \mathcal{M}_{S^n} \otimes \mathcal{M}_X \to \mathcal{M}_X$ denote the projection. Then we may assume that $p_2 \circ \mathcal{M}_F = \mathcal{M}_f$ (equals, not just up to DG homotopy — see Proposition A.2 of the appendix). Since S^n is a *formal* space, there is a quasi-isomorphism of DG algebras $\psi\colon \mathcal{M}_{S^n} \to H^*(S^n;\mathbb{Q})$ and hence a quasi-isomorphism $\psi \otimes 1\colon \mathcal{M}_{S^n} \otimes \mathcal{M}_X \to H^*(S^n;\mathbb{Q}) \otimes \mathcal{M}_X$. Given $\chi \in \mathcal{M}_Y$, we may write

$$(\psi \otimes 1) \circ \mathcal{M}_F(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_F(\chi),$$

thus defining a linear map $\theta_F\colon \mathcal{M}_Y \to \mathcal{M}_X$ that reduces degree by n. That $(\psi \otimes 1) \circ \mathcal{M}_F$ is multiplicative and commutes with differentials yields that θ_F is an $\mathcal{M}_f$-derivation and a $D_{\mathcal{M}_f}$-cycle in $\mathrm{Der}_n(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$. Set $\Phi'_f(\alpha) = \langle\theta_F\rangle$.

To show $\Phi'_f(\alpha)$ is well-defined, suppose that $g_1, g_2\colon S^n \to \mathrm{map}(X, Y; f)$ are homotopic representatives of α with respective adjoint maps $F, G\colon S^n \times X \to Y$. Since the homotopy between g_1 and g_2 is *based*, its adjoint gives a *homotopy under* f *from* F *to* G, in the sense discussed in the appendix. By Proposition A.2, we may assume the minimal models $\mathcal{M}_F$ and $\mathcal{M}_G$ are DG homotopic *as maps over* $\mathcal{M}_f$. That is, we have a homotopy $H\colon \mathcal{M}_Y \to H^*(S^n;\mathbb{Q}) \otimes \Lambda(u, \mathrm{d}u) \otimes \mathcal{M}_X$ with $Q \circ H = ((\psi\otimes 1)\circ\mathcal{M}_F, (\psi\otimes 1)\circ\mathcal{M}_G)$. Here, $Q\colon H^*(S^n;\mathbb{Q})\otimes\Lambda(u, \mathrm{d}u)\otimes\mathcal{M}_X \to H^*(S^n \vee S^n;\mathbb{Q})\otimes\mathcal{M}_X$ is the path object for the projection $p_2\colon H^*(S^n;\mathbb{Q})\otimes\mathcal{M}_X \to \mathcal{M}_X$ described in Example A.1. Modulo the ideal J generated by elements of $H^*(S^n;\mathbb{Q}) \otimes \Lambda(u, \mathrm{d}u)$ of degree greater than $n+1$, we may write

$$H(\chi) \equiv 1 \otimes 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes 1 \otimes \psi_1(\chi) + 1 \otimes u \otimes \psi_2(\chi) + 1 \otimes \mathrm{d}u \otimes \psi_3(\chi).$$

Using the fact that H is both multiplicative and a DG map, as in the previous part, we find that ψ_i defines an $\mathcal{M}_f$-derivation for each i and furthermore that $D(\psi_3) = (-1)^n\psi_2$. Since $Q \circ H = ((\psi \otimes 1) \circ \mathcal{M}_F, (\psi \otimes 1) \circ \mathcal{M}_G)$ and $Q(J) = 0$, it follows that $\theta_F - \theta_G = 2\psi_2$. Hence we have $\langle\theta_F\rangle = \langle\theta_G\rangle$ and so Φ' is well-defined.

Now we show that Φ'_f is a homomorphism. Let $\alpha, \beta \in \pi_n(\mathrm{map}(X, Y; f))$ have adjoints $A, B\colon S^n \times X \to Y$. Let $(A \mid B)_f$ denote the adjoint of $(\alpha \mid \beta)\colon S^n \vee S^n \to \mathrm{map}(X, Y; f)$. That is, if $i_1, i_2\colon S^n \to S^n \vee S^n$ denote the inclusions, then $(A \mid B)_f$ is the map defined by $(A \mid B)_f \circ (i_1 \times 1) = A$ and $(A \mid B)_f \circ (i_2 \times 1) = B$. Let $\psi'\colon \mathcal{M}_{S^n\vee S^n} \to H^*(S^n \vee S^n;\mathbb{Q})$ be a quasi-isomorphism for the formal space $S^n \vee S^n$. Since $(A \mid B)_f$ is a map under f, we may assume by Proposition A.2 that $(\psi' \otimes 1) \circ \mathcal{M}_{(A|B)_f}$ is of the form

$$(\psi' \otimes 1) \circ \mathcal{M}_{(A|B)_f}(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_1(\chi) + t_n \otimes \theta_2(\chi),$$

for $\chi \in \mathcal{M}_Y$. Here we have written $H^*(S^n \vee S^n;\mathbb{Q})$ as $\Lambda(s_n, t_n)/I$.

Now the projection of $(\psi'\otimes 1)\circ\mathcal{M}_{(A|B)_f}$ onto $H^*(S^n;\mathbb{Q})\otimes\mathcal{M}_X$ by $H(i_1)\otimes 1$ is a minimal model for $A\colon S^n \times X \to Y$. Moreover, since the composition $(\alpha \mid \beta) \circ i_1$ is determined up to based homotopy as α, the composition $(A \mid B)_f \circ (i_1 \times 1)$ is determined up to a *homotopy under* f as A. Therefore, $(H(i_1)\otimes 1)\circ(\psi'\otimes 1)\circ\mathcal{M}_{(A|B)_f}\colon \mathcal{M}_Y \to H^*(S^n;\mathbb{Q}) \otimes \mathcal{M}_X$ and $(\psi \otimes 1) \circ \mathcal{M}_A$ are DG homotopic as maps over $\mathcal{M}_f$, by Proposition A.2. By the argument used above to establish that Φ'_f is well-defined, it follows that $\langle\theta_1\rangle = \Phi'_f(\alpha)$ in $H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$. Likewise, we have $\langle\theta_2\rangle = \Phi'_f(\beta)$. Now let $\sigma\colon S^n \to S^n \vee S^n$ denote the usual pinching comultiplication. Then the sum $\alpha + \beta$ is the composition $(\alpha \mid \beta) \circ \sigma$ and its adjoint is $C = (A \mid B)_f \circ (\sigma \times 1)\colon S^n \times X \to Y$. We have $(\psi \otimes 1) \circ \mathcal{M}_C = (H(\sigma) \otimes 1) \circ (\psi' \otimes 1) \circ \mathcal{M}_{(A|B)_f}$. Since $H(\sigma)(s_n) = s_n = H(\sigma)(t_n)$, we have $\Phi'_f(\alpha + \beta) = \langle\theta_1\rangle + \langle\theta_2\rangle = \Phi'_f(\alpha) + \Phi'_f(\beta)$ and Φ'_f is a homomorphism.

Next we show Φ_f is surjective. Denote by $[S^n_{\mathbb{Q}} \times X_{\mathbb{Q}}, Y_{\mathbb{Q}}]^{f_{\mathbb{Q}}}$ the set of homotopy classes of maps $S^n_{\mathbb{Q}} \times X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ under $f_{\mathbb{Q}}$. Using [10, Th.II.3.11] and [20, Th.2.3], we may identify $\pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q}$ with $\pi_n(\mathrm{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}; f_{\mathbb{Q}}))$, and hence with $[S^n_{\mathbb{Q}} \times X_{\mathbb{Q}}, Y_{\mathbb{Q}}]^{f_{\mathbb{Q}}}$. For $\langle\theta\rangle \in H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$ define $\phi(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta(\chi)$ for $\chi \in \mathcal{M}_Y$. Since θ is an $\mathcal{M}_f$-derivation that is a cycle, this defines a DG algebra map $\phi\colon \mathcal{M}_Y \to H^*(S^n;\mathbb{Q}) \otimes \mathcal{M}_X$. Now lift ϕ through the surjective quasi-isomorphism $\psi \otimes 1$ as in [5, Lem.12.4], to obtain a map $\widetilde{\phi}\colon \mathcal{M}_Y \to \mathcal{M}_{S^n} \otimes \mathcal{M}_X$ that satisfies $(\varepsilon \cdot 1) \circ \widetilde{\phi} = \mathcal{M}_f\colon \mathcal{M}_Y \to \mathcal{M}_X$. By the standard correspondence between maps of minimal models and maps of rational spaces, this gives a map $F\colon S^n_{\mathbb{Q}} \times X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ that satisfies $i_2 \circ F \sim f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$. Using, for example, [5, Th.9.7], we can adjust F into a homotopic map $F'\colon S^n_{\mathbb{Q}} \times X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ that satisfies $i_2 \circ F' = f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$, so that F' represents a class of $[S^n_{\mathbb{Q}} \times X_{\mathbb{Q}}, Y_{\mathbb{Q}}]^{f_{\mathbb{Q}}}$. As described at the start of this paragraph, F' corresponds to a homotopy class $\alpha \in \pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q}$. Evidently, we have $\Phi_f(\alpha) = \langle\theta\rangle$.

To show Φ_f is injective it is sufficient to show that $\alpha \in \pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q}$ is zero whenever $\Phi_f(\alpha) = 0$. Using the identification of the previous paragraph, let $G\colon S^n_{\mathbb{Q}} \times X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ be the adjoint map for α. Suppose that

$\theta_G = D(\eta)$ for $\eta \in \mathrm{Der}_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$. Using η, define a map $\Gamma: \mathcal{M}_Y \to H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X \otimes \Lambda(t, \mathrm{d}t)$ by

$$\Gamma(\chi) = 1 \otimes \mathcal{M}_f(\chi) \otimes 1 + s_n \otimes \theta_G(\chi) \otimes (1-t) + s_n \otimes \eta(\chi) \otimes \mathrm{d}t,$$

that is easily checked to be a DG algebra map. Furthermore, it is a DG homotopy – now in the sense used in [5] – from $\mathcal{M}_G$ to the map $E: \mathcal{M}_Y \to H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$ given by $E(\chi) = 1 \otimes \mathcal{M}_f(\chi)$. But this latter map is a Sullivan model of the adjoint of $0 \in \pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q}$. Therefore, the DG homotopy translates into a homotopy between adjoint maps $S^n_{\mathbb{Q}} \times X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ for α and 0. Since a based map from a sphere is null-homotopic if and only if it is based null-homotopic (cf. [21, p. 27]), it follows that $\alpha = 0$ and thus Φ_f is injective.

It is straightforward to check that all faces of the cube commute. $\square$

As a consequence, we retrieve a characterization of the rationalized Gottlieb groups given by Félix and Halperin (see [5, Sec.29(c)]) and extend this characterization to the rationalized evaluation subgroup of a map.

Corollary 2.2. *Let $f: X \to Y$ be a map between simply connected complexes of finite type with X finite. The rationalized nth evaluation subgroup $G_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \cong G_n(Y, X; f) \otimes \mathbb{Q}$ of the map f is isomorphic to the image of the induced homomorphism $H(\varepsilon_*): H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \to H_n(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon))$ for $n \geq 2$. In particular, $G_n(X_{\mathbb{Q}}) \cong G_n(X) \otimes \mathbb{Q}$ is isomorphic to* $\mathrm{im}\{H(\varepsilon_*): H_n(\mathrm{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) \to H_n(\mathrm{Der}(\mathcal{M}_X, \mathbb{Q}; \varepsilon))\}$. $\square$

We observe that taking Y a rational Eilenberg–MacLane space $K(V, n)$, for some (ungraded) vector space V, Theorem 2.1 retrieves its antecedent [23, Th.2] in the rational homotopy setting. Namely, it gives an isomorphism

$$\pi_n(\mathrm{map}(X, K(V, n); f)) \cong H^{m-n}(X; V).$$

We give one further implication of Theorem 2.1. Define an F_0-*space* to be a finite, simply connected complex with finite dimensional rational homotopy (a *rationally elliptic space*) such that $H^{\mathrm{odd}}(X; \mathbb{Q}) = 0$. This type of space features in the following well-known conjecture of Halperin (cf. [5, p. 516]):

Conjecture 2.3. *Suppose X is an F_0-space. Then any fibration $X \to E \to B$ of simply connected spaces is TNCZ, that is, the fibre inclusion $j: X \to E$ induces a surjection on rational cohomology.*

We give an equivalent version of this conjecture in Section 4, below. Here we observe that Theorem 2.1 and the argument given in [8, Cor. 4.6] give an isomorphism

$$\pi_{2n}(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \cong \mathrm{Der}_{2n}(H^*(Y; \mathbb{Q}), H^*(X; \mathbb{Q}); H(f))$$

for $n > 0$ and any map $f: X \to Y$ between F_0-spaces.

3. Derivation spaces and the rationalized G-sequence

In this section, we identify the rationalized long exact homotopy sequences of the maps $f: X \to Y$ and $f_*: \mathrm{map}(X, X; 1) \to \mathrm{map}(X, Y; f)$, and hence the rationalized G-sequence of f. Our identifications flow from the observation that the third term in a long exact sequence of vector spaces is unique up to (non-natural) isomorphism:

Lemma 3.1. *Suppose given diagrams of vector spaces*

$$\begin{tikzcd}
A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\cong"', "\alpha_{n+1}"] & B_{n+1} \arrow[r, "j_{n+1}"] \arrow[d, "\cong"', "\beta_{n+1}"] & C_{n+1} \arrow[r, "k_{n+1}"] \arrow[d, dotted, "\gamma_{n+1}"] & A_n \arrow[r, "i_n"] \arrow[d, "\cong"', "\alpha_n"] & B_n \arrow[d, "\cong"', "\beta_n"] \\
X_{n+1} \arrow[r, "p_{n+1}"'] & Y_{n+1} \arrow[r, "q_{n+1}"'] & Z_{n+1} \arrow[r, "r_{n+1}"'] & X_n \arrow[r, "p_n"'] & Y_n
\end{tikzcd}$$

for each $n \geq 2$. Suppose the rows are exact, each α_n and β_n is an isomorphism, and $\beta_n \circ i_n = p_n \circ \alpha_n$ for each n. Then there exist isomorphisms $\gamma_{n+1}: C_{n+1} \to Z_{n+1}$, for $n \geq 2$, which make the entire ladder commutative.

Proof. The proof is a straightforward diagram chase. $\square$

We next recall the definition of the mapping cone and the long exact homology sequence of a chain map $\phi: A \to B$.

Definition 3.2 (*[21, p. 166]*). Let $\phi\colon A \to B$ be a map of DG vector spaces. Define a DG vector space, called the *mapping cone of* ϕ and denoted by $\mathrm{Rel}_*(\phi)$, as follows: $\mathrm{Rel}_n(\phi) = A_{n-1} \oplus B_n$, with differential δ $(= \delta_\phi)$ of degree -1 given by $\delta(a, b) = (-\mathrm{d}_A(a), \phi(a) + \mathrm{d}_B(b))$. Further, define chain maps $J\colon B_n \to \mathrm{Rel}_n(\phi)$ and $P\colon \mathrm{Rel}_n(\phi) \to A_{n-1}$ by $J(b) = (0, b)$ and $P(a, b) = a$. These give a short exact sequence of chain complexes

$$0 \longrightarrow B_* \xrightarrow{J} \mathrm{Rel}_*(\phi) \xrightarrow{P} A_{*-1} \longrightarrow 0$$

which leads to a long exact sequence in homology

$$\cdots \to H_{n+1}(\mathrm{Rel}(\phi)) \xrightarrow{H(P)} H_n(A) \xrightarrow{H(\phi)} H_n(B) \xrightarrow{H(J)} H_n(\mathrm{Rel}(\phi)) \to \cdots,$$

whose connecting homomorphism is $H(\phi)$. We refer to this sequence as the *long exact homology sequence of* ϕ.

Suppose given maps $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ maps of DG vector spaces such that $\beta \circ \phi = \phi' \circ \alpha$. Then the obvious map $(\alpha, \beta)\colon \mathrm{Rel}_*(\phi) \to \mathrm{Rel}_*(\phi')$ is a chain map that satisfies $(\alpha, \beta) \circ J = J' \circ \beta$ and $\alpha \circ P = P' \circ (\alpha, \beta)$. Thus we obtain a homology ladder by using these maps to map the long exact homology sequence of ϕ to that of ϕ'.

Theorem 3.3. *Let $f\colon X \to Y$ be a map between simply connected CW complexes with X finite. The long exact sequence induced by*

$$f_*\colon \mathrm{map}(X, X; 1) \to \mathrm{map}(X, Y; f)$$

on rational homotopy groups is equivalent to the long exact homology sequence of the map

$$(\mathcal{M}_f)^*\colon \mathrm{Der}_*(\mathcal{M}_X, \mathcal{M}_X; 1) \to \mathrm{Der}_*(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$$

induced by the minimal model $\mathcal{M}_f\colon \mathcal{M}_Y \to \mathcal{M}_X$ of the map $f\colon X \to Y$.

Proof. The result follows directly from Theorem 2.1 and Lemma 3.1. $\square$

We now identify the G-sequence within our current framework. Suppose given a DG algebra map $\phi\colon A \to B$. Starting from this map, we can construct the following commutative square of DG vector spaces:

$$\begin{tikzcd}
\mathrm{Der}_*(B, B; 1) \arrow[r, "\phi^*"] \arrow[d, "\varepsilon_*"'] & \mathrm{Der}_*(A, B; \phi) \arrow[d, "\varepsilon_*"] \\
\mathrm{Der}_*(B, \mathbb{Q}; \varepsilon) \arrow[r, "\widehat{\phi^*}"] & \mathrm{Der}_*(A, \mathbb{Q}; \varepsilon).
\end{tikzcd}$$

In this diagram, ε denotes the augmentation of either A or B, and we have used a decoration to distinguish the lower horizontal map from the upper. On passing to homology and using the naturality of the mapping cone construction, we obtain the following homology ladder ($n \geq 2$):

$$\begin{tikzcd}
\cdots \arrow[r, "H(J)"] & H_{n+1}(\mathrm{Rel}(\phi^*)) \arrow[r, "H(P)"] \arrow[d, "H(\varepsilon_*, \varepsilon_*)"] & H_n(\mathrm{Der}(B, B; 1)) \arrow[r, "H(\phi^*)"] \arrow[d, "H(\varepsilon_*)"] & H_n(\mathrm{Der}(A, B; \phi)) \arrow[d, "H(\varepsilon_*)"] \; \cdots \\
\cdots \arrow[r, "H(J)"] & H_{n+1}(\mathrm{Rel}(\widehat{\phi^*})) \arrow[r, "H(\widehat{P})"] & H_n(\mathrm{Der}(B, \mathbb{Q}; \varepsilon)) \arrow[r, "H(\widehat{\phi^*})"] & H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon)) \; \cdots
\end{tikzcd}$$

Definition 3.4. Suppose $\phi\colon A \to B$ is a map of DG algebras. We define the *evaluation subgroup of* ϕ by

$$G_n(A, B; \phi) = \mathrm{im}\{H(\varepsilon_*)\colon H_n(\mathrm{Der}(A, B; \phi)) \to H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon))\}.$$

In the special case in which $A = B$ and $\phi = 1_B$, we refer to the *Gottlieb group of* B, and use the notation $G_n(B)$. We define the *nth relative evaluation subgroup of* ϕ by

$$G_n^{\mathrm{rel}}(A, B; \phi) = \mathrm{im}\{H(\varepsilon_*, \varepsilon_*)\colon H_n(\mathrm{Rel}(\phi^*)) \to H_n(\mathrm{Rel}(\widehat{\phi^*}))\}.$$

Then the image of the upper long exact sequence in the lower, of the ladder above, gives a (not necessarily exact) sequence

$$\cdots \xrightarrow{H(\widehat{J})} G^{\mathrm{rel}}_{n+1}(A, B; \phi) \xrightarrow{H(\widehat{P})} G_n(B) \xrightarrow{H(\widehat{\phi^*})} G_n(A, B; \phi) \xrightarrow{H(\widehat{J})} \cdots$$

that we terminate in $G_2(A, B; \phi)$. We refer to this sequence as *the G-sequence of the map $\phi: A \to B$.*

All of the above can be applied to the minimal model $\mathcal{M}_f : \mathcal{M}_Y \to \mathcal{M}_X$ of the map $f: X \to Y$. By doing so, and then collecting together previous results, we obtain the following result.

Theorem 3.5. *Let $f: X \to Y$ be a map between simply connected CW complexes with X finite. The rationalization of the G-sequence of the map $f: X \to Y$, as far as the term $G_2(Y, X; f)$, is equivalent to the G-sequence of its Sullivan minimal model $\mathcal{M}_f : \mathcal{M}_Y \to \mathcal{M}_X$, as defined in Definition 3.4.*

Proof. Starting from the cube displayed in the proof of Theorem 2.1, we extend each of the four left-to-right maps into their respective long exact sequences. This is then completed into an equivalence of ladders, by defining isomorphisms γ_n and $\widehat{\gamma_n}$ to give a commutative square

$$\begin{tikzcd}
\pi_n(f_*) \otimes \mathbb{Q} \arrow[r, "\gamma_n"] \arrow[d, "\omega_\#"'] & H_n(\mathrm{Rel}(\phi^*)) \arrow[d, "H(\varepsilon_*, \varepsilon_*)"] \\
\pi_n(f) \otimes \mathbb{Q} \arrow[r, "\widehat{\gamma_n}"] & H_n(\mathrm{Rel}(\widehat{\phi^*}))
\end{tikzcd}$$

for each $n \geq 3$. If these isomorphisms are each defined separately as in Lemma 3.1, using the top and bottom faces of the cube, then a technical problem arises due to the non-natural choice of splittings made there. However, this problem may be surmounted by Lemma 3.6 below.

The result now follows, since the equivalence of ladders restricts to give an equivalence of the corresponding sequences of images. $\square$

Lemma 3.6. *Suppose given diagrams of vector spaces*

$$\begin{tikzcd}
& X_{n+1} \arrow[rr, "p_{n+1}"] \arrow[dd, "t_{n+1}"] & & Y_{n+1} \arrow[rr, "q_{n+1}"] \arrow[dd, "u_{n+1}"] & & Z_{n+1} \arrow[rr, "r_{n+1}"] \arrow[dd, "v_{n+1}"] & & X_n \arrow[rr, "p_n"] \arrow[dd, "t_n"] & & Y_n \arrow[dd, "u_n"] \\
A_{n+1} \arrow[ur, "\cong", "\alpha_{n+1}"'] \arrow[rr, "i_{n+1}"'] \arrow[dd, "f_{n+1}"'] & & B_{n+1} \arrow[ur, "\cong", "\beta_{n+1}"'] \arrow[rr, "j_{n+1}"'] \arrow[dd, "g_{n+1}"] & & C_{n+1} \arrow[ur, dotted, "\gamma_{n+1}"'] \arrow[rr, "k_{n+1}"'] \arrow[dd, "h_{n+1}"] & & A_n \arrow[ur, "\cong", "\alpha_n"'] \arrow[rr, "i_n"'] \arrow[dd, "f_n"] & & B_n \arrow[ur, "\cong", "\beta_n"'] \arrow[dd, "g_n"] & \\
& X'_{n+1} \arrow[rr, "p'_{n+1}"'] & & Y'_{n+1} \arrow[rr, "q'_{n+1}"'] & & Z'_{n+1} \arrow[rr, "r'_{n+1}"'] & & X'_n \arrow[rr, "p'_n"'] & & Y'_n \\
A'_{n+1} \arrow[ur, "\cong", "\alpha'_{n+1}"'] \arrow[rr, "i'_{n+1}"'] & & B'_{n+1} \arrow[ur, "\cong", "\beta'_{n+1}"'] \arrow[rr, "j'_{n+1}"'] & & C'_{n+1} \arrow[ur, dotted, "\gamma'_{n+1}"'] \arrow[rr, "k'_{n+1}"'] & & A'_n \arrow[ur, "\cong", "\alpha'_n"'] \arrow[rr, "i'_n"'] & & B'_n \arrow[ur, "\cong", "\beta'_n"'] &
\end{tikzcd}$$

for $n \geq 2$. Suppose both top and bottom faces satisfy all hypotheses of Lemma 3.1, that front and back ladders are commutative, and that the given internal faces commute, that is, $t_n \circ \alpha_n = \alpha'_n \circ f_n$ and $u_n \circ \beta_n = \beta'_n \circ g_n$. Then isomorphisms γ_{n+1} and γ'_{n+1} exist, for $n \geq 2$, that make the entire diagram commute.

Proof. Again, the proof is a diagram chase and so omitted. $\square$

Remark 3.7. Our results also allow for a description of the long exact rational homotopy sequence of the evaluation fibration $\mathrm{map}_*(X, Y; f) \to \mathrm{map}(X, Y; f) \to Y$. Starting with the short exact sequence of DG vector spaces

$$0 \longrightarrow \ker(\varepsilon_*) \xrightarrow{I} \mathrm{Der}_*(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f) \xrightarrow{\varepsilon_*} \mathrm{Der}_*(\mathcal{M}_Y, \mathbb{Q}; \varepsilon) \longrightarrow 0$$

we obtain a long exact sequence on homology

$$\cdots \to H_n(\ker(\varepsilon_*)) \to H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \to H_n(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)) \to \cdots$$

This sequence is equivalent to the long exact rational homotopy sequence of the evaluation fibration. In particular, we have $\pi_n(\mathrm{map}_*(X, Y; f)) \otimes \mathbb{Q} \cong H_n(\ker(\varepsilon_*))$ for $n \geq 2$.

4. Examples and further consequences

The first example of a non-exact G-sequence was given in [12]; further examples were given in [17] and [18]. To illustrate the effectiveness of the framework established above, we give a composite example in which the G-sequence of a map fails to be exact (even after rationalization) at each of the three types of term that occur.

Example 4.1. Let $f = (f_1, f_2): \mathbb{H}P^2 \to S^8 \times \mathbb{H}P^4$ be the map with coordinate functions $f_1: \mathbb{H}P^2 \to S^8$ obtained by pinching out the bottom cell and $f_2: \mathbb{H}P^2 \to \mathbb{H}P^4$ the inclusion. Denote $\mathbb{H}P^2$ by X and $S^8 \times \mathbb{H}P^4$ by Y, thus $f: X \to Y$. A straightforward, if lengthy, computation using Theorem 3.5 shows that the G-sequence of f is non-exact at the terms $G_4(Y, X; f)$, $G_8^{\mathrm{rel}}(Y, X; f)$, and $G_{11}(X)$. We give the details for the term $G_4(Y, X; f)$.

First, $\mathcal{M}_X = \Lambda(x_4, x_{11})$, with differential given on generators by $\mathrm{d}(x_4) = 0$, and $\mathrm{d}(x_{11}) = x_4^3$, and $\mathcal{M}_Y = \Lambda(y_8, y_{15}, y_4, y_{19})$ with differential $\mathrm{d}(y_8) = 0$, $\mathrm{d}(y_{15}) = y_8^2$, $\mathrm{d}(y_4) = 0$, and $\mathrm{d}(y_{19}) = y_4^5$. In both models, subscripts denote degrees. Then the Sullivan model of f, which we denote by $\phi: \mathcal{M}_Y \to \mathcal{M}_X$, is given on generators by $\phi(y_8) = x_4^2$, $\phi(y_{15}) = x_4 x_{11}$, $\phi(y_4) = x_4$, and $\phi(y_{19}) = x_4^2 x_{11}$.

Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Der}_4(\mathcal{M}_X, \mathcal{M}_X; 1) \arrow[r, "\phi^*"] \arrow[d, "\varepsilon_*"] & \mathrm{Der}_4(\mathcal{M}_Y, \mathcal{M}_X; \phi) \arrow[r, "J"] \arrow[d, "\varepsilon_*"] & \mathrm{Rel}_4(\phi^*) \arrow[d, "(\varepsilon_*, \varepsilon_*)"] \\
\mathrm{Der}_4(\mathcal{M}_X, \mathbb{Q}; \varepsilon) \arrow[r, "\widehat{\phi^*}"] & \mathrm{Der}_4(\mathcal{M}_Y, \mathbb{Q}; \varepsilon) \arrow[r, "\widehat{J}"] & \mathrm{Rel}_4(\widehat{\phi^*})
\end{tikzcd}$$

Define a ϕ-derivation $\theta \in \mathrm{Der}_4(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ by setting $\theta(y_{19}) = 5x_4x_{11}$ and $\theta(y_4) = 1$. It is direct to check that θ is a cocycle. Under

$$H(\varepsilon_*): H_4(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \phi)) \to H_4(\mathrm{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)),$$

we have $H(\varepsilon_*)(\langle\theta\rangle) = \langle y_4^* \rangle \neq 0$ where $y_4^* \in \mathrm{Der}(\mathcal{M}, \mathbb{Q}; \varepsilon)$ denotes the dual of y_4. Since $\langle y_4^* \rangle = H(\widehat{\phi^*})(\langle x_4^* \rangle)$, it follows that $H(\widehat{J})(\langle y_4^* \rangle) = 0$. However, $G_4(\mathcal{M}_X) = 0$. Therefore, $\langle y_4^* \rangle \in G_4(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ is a non-zero element in the kernel of $H(\widehat{J}): G_4(\mathcal{M}_Y, \mathcal{M}_X; \phi) \to G_4^{\mathrm{rel}}(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ that is not in the image of $H(\widehat{\phi^*}): G_4(\mathcal{M}_X) \to G_4(\mathcal{M}_Y, \mathcal{M}_X; \phi)$.

We next give a sample result concerning the exactness of the G-sequence. Given a space X, define a linear map of degree zero

$$\varphi_X: H_*(\mathrm{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) \to \mathrm{Der}_*(H^*(X; \mathbb{Q}), H^*(X; \mathbb{Q}); 1)$$

by the rule $\varphi_X(\langle\theta\rangle)(\langle\chi\rangle) = \langle\theta(\chi)\rangle$, for θ a cycle in $\mathrm{Der}_*(\mathcal{M}_X, \mathcal{M}_X; 1)$ and χ a cocycle in $\mathcal{M}_X$. It is straightforward to check that φ_X is well-defined. (See [8, Prop.1.6]; in fact, φ_X is a morphism of graded Lie algebras.) We note that the map φ_X makes an appearance, in a completely different context, in the work of Belegradek and Kapovitch [2].

Theorem 4.2. *Let X be a finite complex for which the map φ_X defined above is trivial and let Y be a rational H-space. Then the rationalized G-sequence for any map $f: X \to Y$ is exact at the term $G_n(X) \otimes \mathbb{Q}$.*

Proof. Since Y is a rational H-space, its minimal model $\mathcal{M}_Y \cong H^*(Y; \mathbb{Q})$ has trivial differential. Let $\phi: H^*(Y; \mathbb{Q}) \to \mathcal{M}_X$ denote the minimal model of f. For a derivation $\theta \in \mathrm{Der}_*(H^*(Y; \mathbb{Q}), \mathcal{M}_X; \phi)$, we have $D(\theta) = \pm \mathrm{d}_X \theta$. Using this observation, we obtain a map

$$\mu: H_*(\mathrm{Der}_*(H^*(Y; \mathbb{Q}), \mathcal{M}_X; \phi)) \to \mathrm{Der}_*(H^*(Y; \mathbb{Q}), H^*(X; \mathbb{Q}); H(\phi)),$$

defined by $\mu([\theta])(\chi) = [\theta(\chi)]$. Using the preceding observation, together with the free-ness of $H^*(Y;\mathbb{Q})$, it is straightforward to check that μ is an isomorphism. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} H_*(\mathrm{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) & \xrightarrow{H(\phi^*)} & H_*(\mathrm{Der}(H^*(Y;\mathbb{Q}), \mathcal{M}_X; \phi)) \\ \Big\downarrow{\scriptstyle \varphi_X} & & \cong\Big\downarrow{\scriptstyle \mu} \\ \mathrm{Der}_*(H^*(X;\mathbb{Q}), H^*(X;\mathbb{Q}); 1) & \xrightarrow[(H(\phi))^*]{} & \mathrm{Der}_*(H^*(Y;\mathbb{Q}), H^*(X;\mathbb{Q}); H(\phi)) \end{array}$$

Therefore, the assumption that $\varphi_X = 0$ implies that the top map $H(\phi^*)$ in the above diagram is zero. A straightforward diagram chase using the homology ladder that defines the rationalized G-sequence now gives the result. □

We conclude with a connection between the rationalized G-sequence and the conjecture of Halperin, Conjecture 2.3, mentioned above.

Theorem 4.3. *Let $X \xrightarrow{j} E \xrightarrow{p} S^{2r+1}$ be any fibration with X an F_0-space. The following are equivalent:*

(1) *The fibration is TNCZ, that is, $H(j)\colon H^*(E;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is surjective.*
(2) *The rationalized G-sequence of the fibre inclusion reduces to split short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & G_n(X)\otimes\mathbb{Q} \arrow[r, "(j)_\#"] & G_n(E,X;j)\otimes\mathbb{Q} \arrow[r, "(p)_\#"'] & \pi_n(S^{2r+1})\otimes\mathbb{Q} \arrow[l, bend right, "(i_2)_\#"'] \arrow[r] & 0 \end{tikzcd}$$

for each $n \geq 2$.

Proof. By [13, Th.2.3], (1) implies the fibration $X \to E \to S^{2n+1}$ is rationally fibre-homotopy equivalent to the product fibration. The implication (1) ⇒ (2) thus follows from the fact that the G-sequence of the inclusion of a summand of a product splits as in (2) (see [24, Cor.13]).

We prove (2) ⇒ (1). Suppose the fibration $X \to E \to S^{2r+1}$ has minimal model

$$\Lambda(u) \xrightarrow{i} \Lambda(u)\otimes\Lambda V, \quad D \xrightarrow{\pi} (\Lambda V, \mathrm{d}),$$

where i denotes the inclusion $i(u) = u\otimes 1$ and π is the projection. By hypothesis, we have that $p_\#\colon G_{2r+1}(E,X;j)\otimes\mathbb{Q} \to \pi_{2r+1}(S^{2r+1})\otimes\mathbb{Q}$ is onto. When translated into our derivation setting, this gives the existence of a π-derivation $\psi \in \mathrm{Der}_{2r+1}(\Lambda(u)\otimes\Lambda V, \Lambda V; \pi)$ that is a cocycle, and that satisfies $\psi(u) = 1$. Using this ψ, define a linear map $\Phi\colon \Lambda(u)\otimes\Lambda V \to \Lambda(u)\otimes\Lambda V$ by setting $\Phi(a+ub) = a + ub + u\psi(a)$ for a typical element $a + ub \in \Lambda(u)\otimes\Lambda V$. We claim that Φ is actually a DG algebra isomorphism $(\Lambda(u)\otimes\Lambda V, D) \to (\Lambda(u)\otimes\Lambda V, \mathrm{d})$. First, it is easy to check that Φ is an algebra map using the fact that ψ is a derivation. Similarly, the fact that Φ commutes with differentials, that is, that $\Phi D = \mathrm{d}\Phi$, follows easily from the fact that ψ is a cocycle. Finally, it is evident that Φ is an isomorphism, since we have $\Phi(u) = u$ and $\pi\circ\Phi = \pi$. Therefore, $\Phi\colon (\Lambda(u)\otimes\Lambda V, D) \to (\Lambda(u)\otimes\Lambda V, \mathrm{d})$ is a DG isomorphism and the fibration is rationally trivial. □

This leads to the following equivalent phrasing of Conjecture 2.3:

Corollary 4.4. *Let X be an F_0-space. Then X satisfies Conjecture 2.3 if and only if the G-sequence of the fibre inclusion in every fibration of the form $X \to E \to S^{2n+1}$ decomposes into split short exact sequences as in (2) of Theorem 4.3.*

Proof. By [16, Lem.2.5], Conjecture 2.3 is equivalent to the collapsing of the rational Serre spectral sequence for all fibrations with fibre X and base an odd sphere. □

Appendix A. Some DG algebra homotopy theory

In this appendix we carefully justify a result from DG algebra homotopy theory that is used in a crucial way to establish Theorem 2.1. Since it is a technical appendix, we rely on a greater degree of familiarity with techniques from rational homotopy theory. We use [1] and [5] as general references here. In what follows, $A^*(-)$ denotes Sullivan's

PL forms functor. For maps of DG algebras, we use a double-headed arrow to denote a surjection and $\simeq$ to denote a quasi-isomorphism. We emphasize that diagrams commute strictly, and not just up to homotopy. Indeed, it is precisely this point that calls for the careful treatment of this appendix.

Suppose given a cofibration $i: X \to Z$ and a map $f: X \to Y$. Then $F: Z \to Y$ is called a *map under* f if $F \circ i = f$. Two such maps F and F' are *homotopic under* f if there is a homotopy from F to F' that is a map under f at each stage. In the setting of DG algebras, suppose given a surjection $p: C \to B$ and a map $g: A \to B$. Then a map $G: A \to C$ is called a *map over* g if $p \circ G = g$. The pullback

$$\begin{tikzcd} C \oplus_B C \arrow[r, "p_2"] \arrow[d, "p_1"'] & C \arrow[d, "p"] \\ C \arrow[r, "p"'] & B \end{tikzcd} \tag{2}$$

defines both a diagonal $\Delta: C \to C \oplus_B C$ and, whenever we have maps $G, G': A \to C$ over g, a map $(G, G'): A \to C \oplus_B C$. A fact that we use frequently here is that any map of DG algebras may be factored as a quasi-isomorphism followed by a surjection (see [5, Lem.12.5 *et seq.*]). See [1] for the following terminology. Given such a factorization of Δ

$$C \xrightarrow{\simeq} P_B C \overset{q}{\twoheadrightarrow} C \oplus_B C,$$

we say that q is a *path object for* p. Then maps G and G' are *homotopic over* g if there is a *homotopy over* g, that is, a map $H: A \to P_B C$, such that $q \circ H = (G, G')$. By dualizing the first part of [1, II.2.2] we see that, if A is minimal, then any convenient path object for p may be used. In the proof of Theorem 2.1, we use the following particular choice of path object.

Example A.1. Given the projection $p_2: H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X \to \mathcal{M}_X$ and a map $\mathcal{M}_f: \mathcal{M}_Y \to \mathcal{M}_X$, we consider maps $\mathcal{M}_Y \to H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$ over $\mathcal{M}_f$. Write $H^*(S^n; \mathbb{Q})$ as $\Lambda(s_n)/I$ and $H^*(S^n \vee S^n; \mathbb{Q})$ as $\Lambda(s_n, s_n')/I$ where, in both cases, I denotes the ideal generated by elements of degree $2n$. Then we may write the pullback $H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X \oplus_{\mathcal{M}_X} H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$ as $H^*(S^n \vee S^n; \mathbb{Q}) \otimes \mathcal{M}_X$ and the diagonal as $\Delta = \sigma \otimes 1$ where $\sigma(s_n) = s_n + s_n'$. Let $\Lambda(u, \mathrm{d}u)$ denote the acyclic DG algebra generated by u in degree n with differential given by $\mathrm{d}(u) = \mathrm{d}u$. Then the diagonal factors as

$$H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X \xrightarrow[\simeq]{j} H^*(S^n; \mathbb{Q}) \otimes \Lambda(u, \mathrm{d}u) \otimes \mathcal{M}_X \overset{Q}{\twoheadrightarrow} H^*(S^n \vee S^n; \mathbb{Q}) \otimes \mathcal{M}_X$$

with j the obvious inclusion and $Q = \bar{\sigma} \otimes 1$, where $\bar{\sigma}(s_n \otimes 1) = s_n + s_n'$, $\bar{\sigma}(1 \otimes u) = s_n - s_n'$ and $\bar{\sigma}(1 \otimes \mathrm{d}u) = 0$. Since j is a quasi-isomorphism and Q a surjection, we may choose Q as a path object for the projection p_2.

For our proof of Theorem 2.1, we want two special cases of the following result:

Proposition A.2. *Let $i: U \to Z$ be a cofibration that admits a retraction. Let $f: U \to Y$ be a fixed map. Choose and fix a minimal model $\mathcal{M}_f: \mathcal{M}_Y \to \mathcal{M}_X$ for f. Then each map $F: Z \to Y$ under f has a minimal model $\mathcal{M}_F: \mathcal{M}_Y \to \mathcal{M}_Z$ that is a map over $\mathcal{M}_f$. If F and F' are homotopic under f, then $\mathcal{M}_F$ and $\mathcal{M}_{F'}$ are DG homotopic over $\mathcal{M}_f$.*

Specifically, we apply this result to the cases in which $i = i_2: X \to S^n \times X$ and $i = (i_j \times 1): S^n \times X \to (S^n \vee S^n) \times X$ with $j = 1, 2$.

Some of the details needed for our proof appear in [1], particularly in sections II.1 and II.2, although the material there must be dualized to the setting of the "fibration category" of DG algebras before we may use it. After dualizing, we may specialize those results to the context of DG algebras using the dictionary "fibration" $\equiv$ surjection, "weak equivalence" $\equiv$ quasi-isomorphism, and "fibrant" $\equiv$ minimal model.

We begin by showing that the Sullivan functor translates the relation of homotopy under into that of DG homotopy over. This step does not require any hypothesis on the cofibration $i: X \to Z$.

Lemma A.3. *Suppose given a cofibration $i: X \to Z$, a map $f: X \to Y$, and maps $F, F': Z \to Y$ homotopic under f. Let $\eta_Y: \mathcal{M}_Y \to A^*(Y)$ be a minimal model. Then $A^*(F) \circ \eta_Y, A^*(F') \circ \eta_Y: \mathcal{M}_Y \to A^*(Z)$ are homotopic over $A^*(f) \circ \eta_Y: \mathcal{M}_Y \to A^*(X)$.*

Proof. Consider the diagrams

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "i"'] & Z \arrow[d, "i_2"] \\ Z \arrow[r, "i_1"'] & Z \cup_X Z \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A^*(Z) \oplus_{A^*(X)} A^*(Z) \arrow[r, "p_2"] \arrow[d, "p_1"'] & A^*(Z) \arrow[d, "A^*(i)"] \\ A^*(Z) \arrow[r, "A^*(i)"'] & A^*(X). \end{tikzcd}$$

with the first a topological pushout and the second a DG algebra pullback. The pushout defines the folding map $\nabla = (1 \mid 1): Z \cup_X Z \to Z$ and the map $(F \mid F'): Z \cup_X Z \to Y$. The pullback, together with the commutative square that results from applying the Sullivan functor A^* to the pushout, defines the map $\omega: A^*(Z \cup_X Z) \to A^*(Z) \oplus_{A^*(X)} A^*(Z)$. This map satisfies $\omega \circ A^*(F \mid F') = (A^*(F), A^*(F')): A^*(Y) \to A^*(Z) \oplus_{A^*(X)} A^*(Z)$ and $\omega \circ A^*(\nabla) = \Delta: A^*(Z) \to A^*(Z) \oplus_{A^*(X)} A^*(Z)$. Factor ∇ as a cofibration followed by a homotopy equivalence $h \circ k: Z \cup_X Z \to I_X Z \to Z$, to yield a cylinder object k for i. If F and F' are homotopic under f, then there is a homotopy $H: I_X Z \to Y$ under f with $H \circ k = (F \mid F'): Z \cup_X Z \to Y$ (cf. [1, I.1.6]). To prove the result, we seek a map $H': \mathcal{M}_Y \to P_{A^*(X)} A^*(Z)$ such that $Q \circ H' = (A^*(F) \circ \eta_Y, A^*(F') \circ \eta_Y): \mathcal{M}_Y \to A^*(Z) \oplus_{A^*(X)} A^*(Z)$ and where Q is a suitable path object for $A^*(i)$. To this end, use $A^*(H)$ and ω to construct the following commutative diagram of solid arrows:

$$\begin{tikzcd} & A^*(Z) \arrow[rr, "\simeq"] \arrow[d, "\simeq"'] \arrow[r, dotted, "\simeq"] & & P_{A^*(X)} A^*(Z) \arrow[d, "Q"] \\ & & \widetilde{A} \arrow[ur, dotted, "\psi"'] \arrow[dl, dotted, two heads, "\simeq"', "\phi"] & \\ \mathcal{M}_Y \arrow[r, "A^*(H) \circ \eta_Y"'] & A^*(I_X Z) \arrow[rr, "\omega \circ A^*(k)"'] & & A^*(Z) \oplus_{A^*(X)} A^*(Z) \end{tikzcd} \tag{3}$$

Here, the top and right arrows are a factorization of Δ as a quasi-isomorphism followed by a surjection, so that Q is a path object for $A^*(i)$. The square commutes because of the identity $\omega \circ A^*(F \mid F') = (A^*(F), A^*(F'))$ noted above. Now we may fill in the dotted arrows, so that all parts commute, by the (dual of the) "weak lifting lemma" of [1, II.1.10]. Then lift $A^*(H) \circ \eta_Y$ through the surjective quasi-isomorphism ϕ, using the ordinary lifting lemma (this is the dual of [1, II.1.6]; see [5, Lem.12.4]). The composition of this lift followed by ψ gives the desired factorization through Q of $(A^*(F) \circ \eta_Y, A^*(F') \circ \eta_Y)$. $\square$

Now we turn to the passage to minimal models. In general we use $\eta_X: \mathcal{M}_X \to A^*(X)$ to denote the minimal model of a space X. For the minimal model of a map $f: X \to Y$, we set the following standard notation (cf. [5, Sec.12(c)]). A minimal model $\eta_X: \mathcal{M}_X \to A^*(X)$ may be factored as

$$\mathcal{M}_X \xrightarrow[\simeq]{\alpha_X} \mathcal{M}_X \otimes E(A^*(X)) \xrightarrow{\gamma_X} A^*(X).$$

The notation $E(A)$ for a DG algebra A denotes the acyclic DG algebra $\Lambda(W, DW)$ with W isomorphic to the underlying module of A. The quasi-isomorphism α_X is simply the inclusion of $\mathcal{M}_X$ as a DG subalgebra, and the surjection γ_X is $\eta_X \cdot \sigma$, where σ denotes the extension of the identity to a map $E(A^*(X)) \to A^*(X)$. Since η_X itself is a quasi-isomorphism, so too is γ_X. Then we may lift $A^*(f) \circ \eta_Y$ through γ_X to obtain a map $\phi_f: \mathcal{M}_Y \to \mathcal{M}_X \otimes E(A^*(X))$ that satisfies $\gamma_X \circ \phi_f = A^*(f) \circ \eta_Y$. The inclusion α_X admits the obvious retraction $\beta_X = 1 \cdot \epsilon: \mathcal{M}_X \otimes E(A^*(X)) \to \mathcal{M}_X$, which also is a quasi-isomorphism. We take the minimal model of f as $\mathcal{M}_f = \beta_X \circ \phi_f$.

We will use the following result of Baues. For a given DG algebra map $u: U \to Y$, we denote the set of DG homotopy (over u) classes of maps $U \to X$ over u by $[U, X]_u$.

Proposition A.4 (*Baues*). *Suppose given a commutative diagram of solid arrows*

$$\begin{tikzcd} U \arrow[r, dotted] \arrow[dr, "u"'] & X \arrow[r, "g", "\simeq"'] \arrow[d] & A \arrow[d] \\ & Y \arrow[r, "\simeq", "g'"'] & B \end{tikzcd}$$

with U minimal. Then $g_*\colon [U, X]_u \to [U, A]_{g' \circ u}$ is a bijection.

Proof. Dualize the arguments given for the "f^*" part of Lemma II.2.9, and Proposition 2.11 of [1]. Note that the result as stated actually holds in any "fibration category" and we specialize to obtain the result for DG algebras using the dictionary mentioned earlier. Note also that this extends the well-known absolute case of this result (see, e.g., [5, Prop.12.9]). □

Proof of Proposition A.2. Suppose $r\colon Z \to U$ is a retraction of $i\colon U \to Z$. Construct a K-S model $i_X\colon \mathcal{M}_X \to (\mathcal{M}_X \otimes \Lambda(V), \mathrm{d})$ for the map $A^*(r) \circ \eta_X\colon \mathcal{M}_X \to A^*(Z)$. Here, i_X denotes the inclusion of $\mathcal{M}_X$ as a DG subalgebra and the construction yields a quasi-isomorphism $\eta\colon \mathcal{M}_X \otimes \Lambda(V) \to A^*(Z)$ that satisfies $\eta \circ i_X = A^*(r) \circ \eta_X$. As a retraction, r induces a surjection on rational homotopy groups and it follows that the differential d must be decomposable. That is, we may identify $\mathcal{M}_Z$ with $(\mathcal{M}_X \otimes \Lambda(V), \mathrm{d})$ and hence $\mathcal{M}_r$ with i_X. Furthermore, r induces an injection on rational cohomology. After a possible change of generators in V, therefore, we may assume that the ideal in $\mathcal{M}_X \otimes \Lambda(V)$ generated by V is d-stable. This results in a commutative diagram

$$\begin{tikzcd} \mathcal{M}_X \arrow[r, "i_X"] \arrow[d, "\eta_X"', "\simeq"] & (\mathcal{M}_X \otimes \Lambda(V), \mathrm{d}) \arrow[r, "1\cdot\epsilon"] \arrow[d, "\eta"', "\simeq"] & \mathcal{M}_X \arrow[d, "\eta_X"', "\simeq"] \\ A^*(X) \arrow[r, "A^*(r)"'] & A^*(Z) \arrow[r, "A^*(i)"'] & A^*(X) \end{tikzcd}$$

of minimal models, in which we may identify η with η_Z, i_X with $\mathcal{M}_r$ and the projection $1 \cdot \epsilon$ with $\mathcal{M}_i$.

Now we may factor η_Z and η_X as described above and obtain a diagram of solid arrows

$$\begin{tikzcd} \mathcal{M}_Z \arrow[r, "\simeq", "\alpha_Z"'] \arrow[d, "\mathcal{M}_i"'] & \mathcal{M}_Z \otimes E(A^*(Z)) \arrow[r, "\gamma_Z", "\simeq"'] \arrow[d, "\mathcal{M}_i \otimes E(A^*(i))"'] \arrow[l, bend right, dotted, "\beta_Z"'] & A^*(Z) \arrow[d, "A^*(i)"'] \\ \mathcal{M}_X \arrow[r, "\alpha_X", "\simeq"'] & \mathcal{M}_X \otimes E(A^*(X)) \arrow[r, "\simeq", "\gamma_X"'] \arrow[l, bend left, dotted, "\beta_X"] & A^*(X) \end{tikzcd} \tag{4}$$

that is (strictly) commutative. Furthermore, the retractions β_Z and β_X of α_Z and α_X, respectively, evidently satisfy $\mathcal{M}_i \circ \beta_Z = \beta_X \circ \mathcal{M}_i \otimes E(A^*(i))$.

Applying Proposition A.4 to the right-hand part of (4) yields a bijection

$$(\gamma_Z)_*\colon [\mathcal{M}_Y, \mathcal{M}_Z \otimes E(A^*(Z))]_{\phi_f} \to [\mathcal{M}_Y, A^*(Z)]_{A^*(i)\circ\eta_Y}.$$

Likewise, from the left-hand commutative square (with β_Z and β_X) we obtain a bijection $(\beta_Z)_*\colon [\mathcal{M}_Y, \mathcal{M}_Z \otimes E(A^*(Z))]_{\phi_f} \to [\mathcal{M}_Y, \mathcal{M}_Z]_{\mathcal{M}_i}$. Combining these bijections with Lemma A.3, we obtain a map

$$[Z, Y]^f \xrightarrow{(\eta_Y)^* \circ A^*(-)} [\mathcal{M}_Y, A^*(Z)]_{A^*(i)\circ\eta_Y} \xrightarrow[\cong]{(\beta_Z)_* \circ ((\gamma_Z)_*)^{-1}} [\mathcal{M}_Y, \mathcal{M}_Z]_{\mathcal{M}_i}.$$

By our conventions, we have $\gamma_X \circ \phi_F = \eta_Y \circ A^*(F)$ and $\beta_X \circ \phi_F = \mathcal{M}_F$. The result follows. □

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